Part II : Residence time

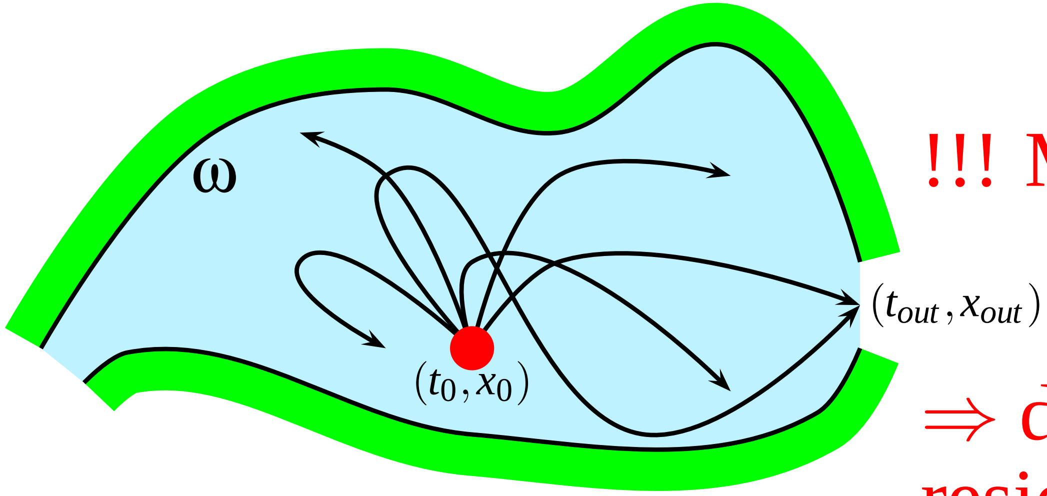
Residence time

Widely used concept in environmental studies :

9814 references found in the Elsevier Catalog
(Science Direct - Environmental Science category)
over the last 10 years !

Very appealing concept to biologists
and decision makers !

Lagrangian approach



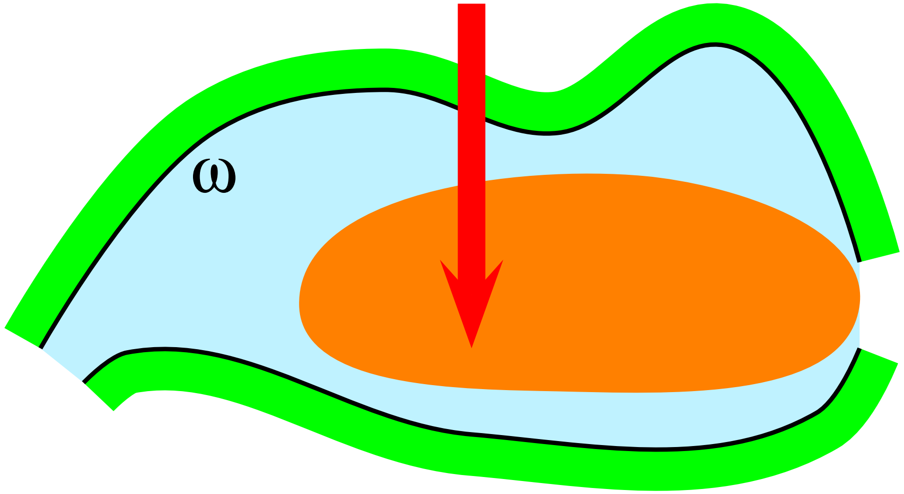
!!! Mixing !!!

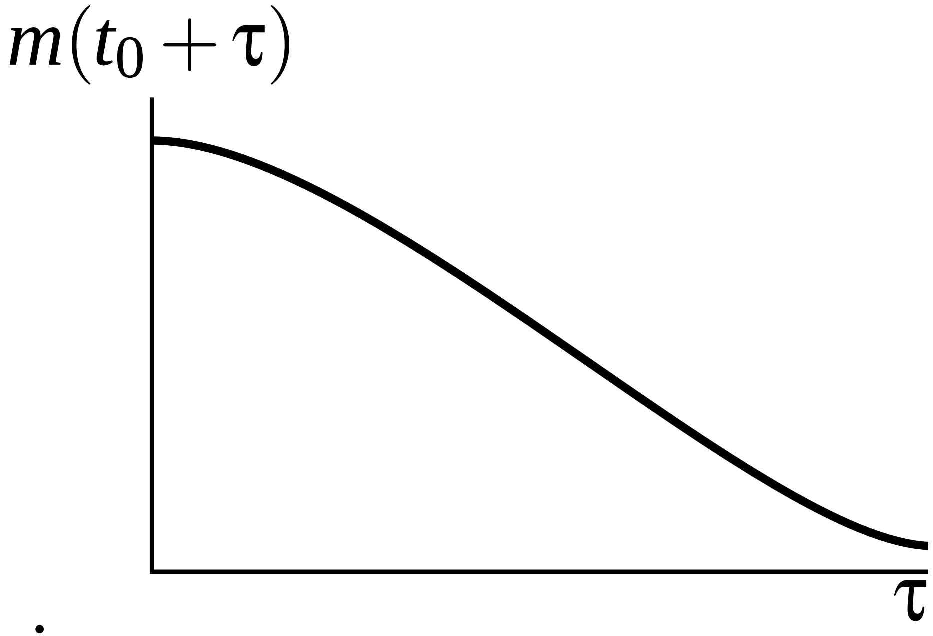
⇒ distribution of residence times

- Define the control domain ω
- Introduce a particle at (t_0, x_0)
- Compute / observe the path of this particle and register its exit time (t_{out}, x_{out})

$$\text{Residence time at } (t_0, x_0) = t_{out} - t_0$$

Eulerian approach

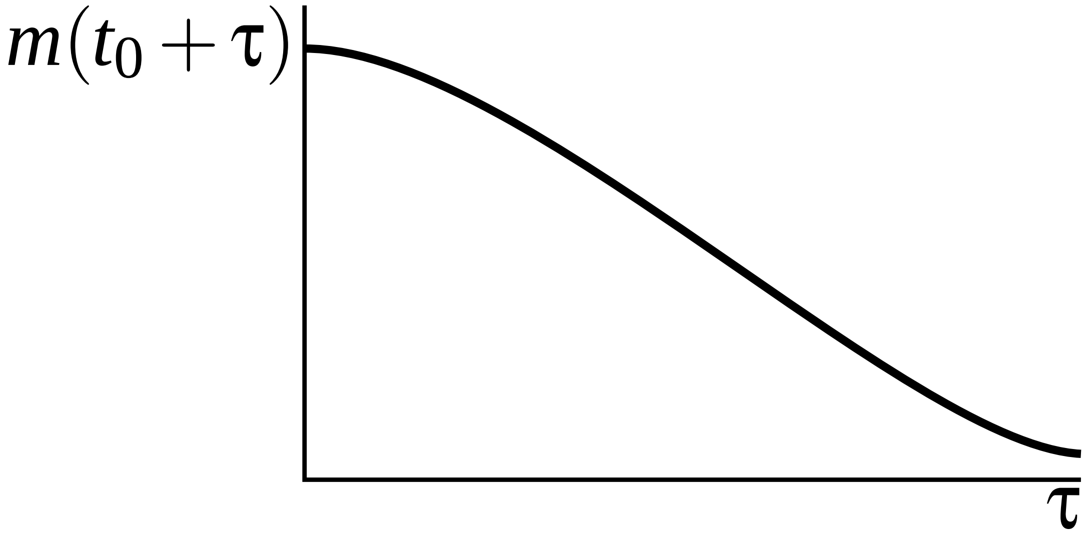




- Define the control domain ω
- Introduce a unit discharge at (t_0, x_0)
- Follow the fate of the tracer and monitor the mass $m(t_0+\tau)$ remaining in ω

$\Rightarrow$ Cumulative distribution of RT

Mean residence time



$\frac{m(t_0+\tau)}{m(t_0)}$ = Fraction of the initial release with a RT larger than τ

Mean residence time $\bar{\theta} = \frac{1}{m(t_0)} \int_0^{\infty} m(t_0+\tau)\,d\tau$

Forward Eulerian procedure

Solve

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \nabla \cdot \left[\mathbf{K} \cdot \nabla C \right] + Q^c \\ C(t_0, x) = \delta(x - x_0) \end{cases}$$

and compute

$$m(t; t_0, x_0) = \iiint_\omega C \; dV$$

But. . . Open boundary conditions ?

BC - version 1

Residence time = time required to leave the control domain **for the first time**

(Bolin and Rodhe, 1973; Takeoka, 1984)

Lagrangian approach : discard particles when they leave the control domain (Lagrangian approach)

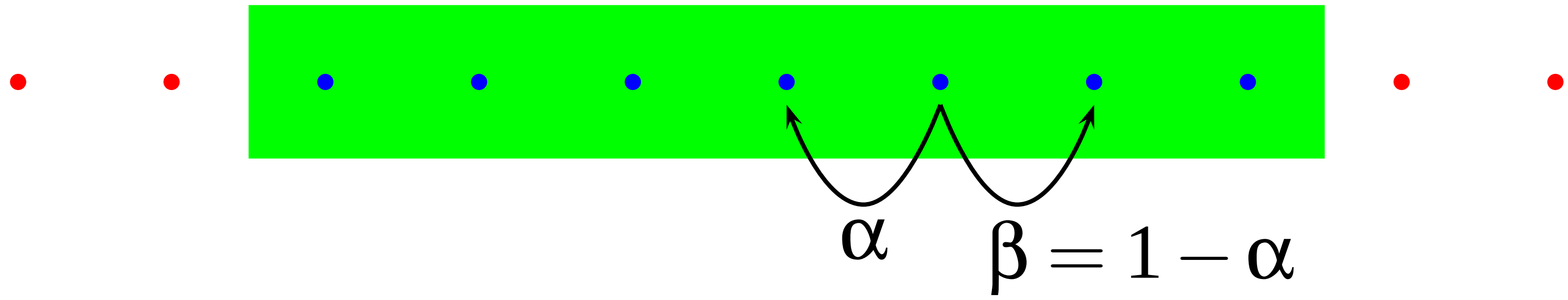
Eulerian approach : put $C = 0$ at both inflow and outflow boundaries of the control domain (if diffusion $\neq 0$)

$\Rightarrow \overline{\theta} = 0$ at the open boundary.

(Delhez & Deleersnijder, 2006. *Ocean Dynamics)*

Asymmetric Random Walk

$$\omega = (-L, L) \cap \mathbb{Z}$$



$p_i^{(n)}$: probability distribution of particles at time t_n.

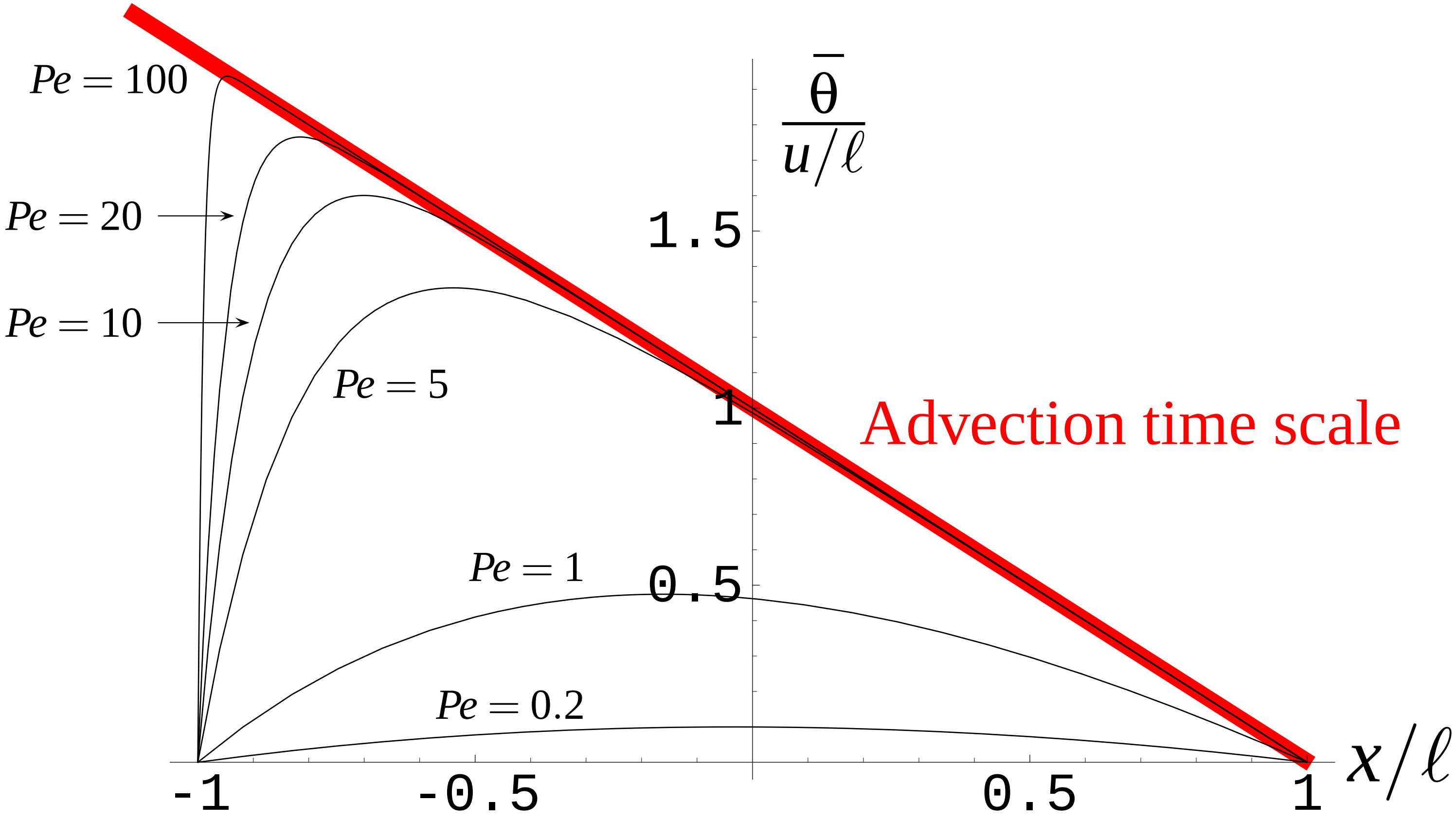
$$p_i^{(n+1)} = \beta\, p_{i-1}^{(n)} + \alpha\, p_{i+1}^{(n)}, \qquad i \in (-L, L) \cap \mathbb{Z}$$

$$p_{-L}^{(n+1)} = \alpha\, p_{-L+1}^{(n)}, \qquad p_L^{(n+1)} = \beta\, p_{L-1}^{(n)}$$

Equivalent to $p_{L+1}^{(n)} = p_{-L-1}^{(n)} = 0$, i.e. $C = 0$ outside ω

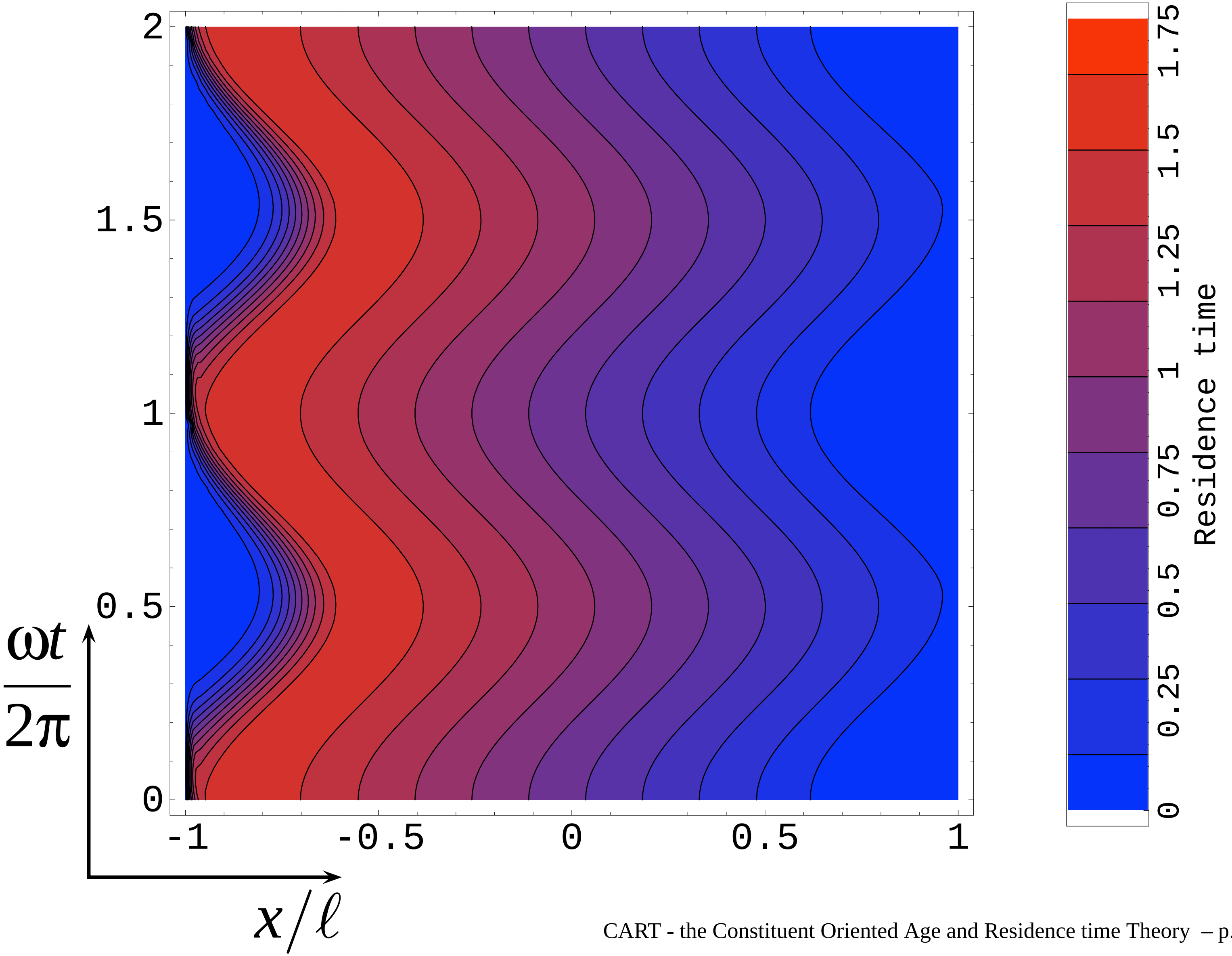
Boundary layers of the RT (1)

- 1D infinite domain $x \in (-\infty, +\infty)$
- $\omega = [-\ell, +\ell]$
- Constant and uniform u and κ (and $Pe = u\ell/\kappa$)



Boundary layers of the RT (2)

Idem with tidal (1 m/s) + residual (0.1 m/s) flow



Exposure time

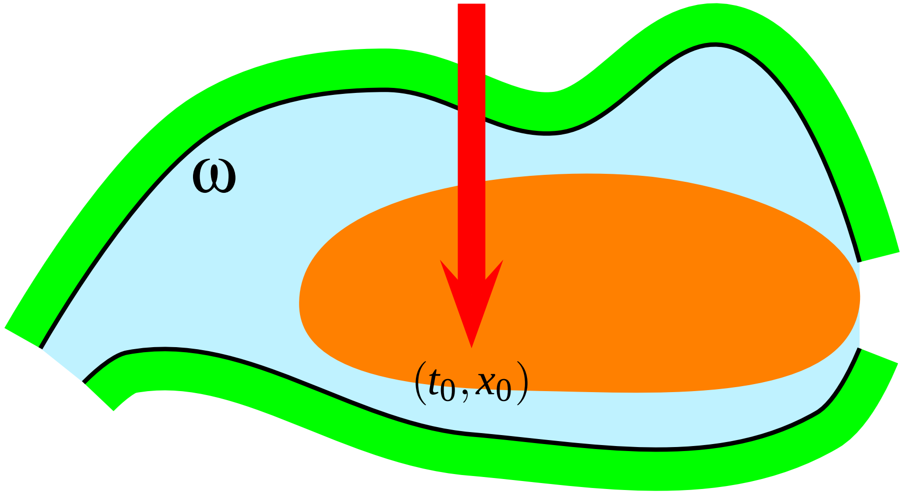
Residence time = time required to leave the control domain **for the first time**

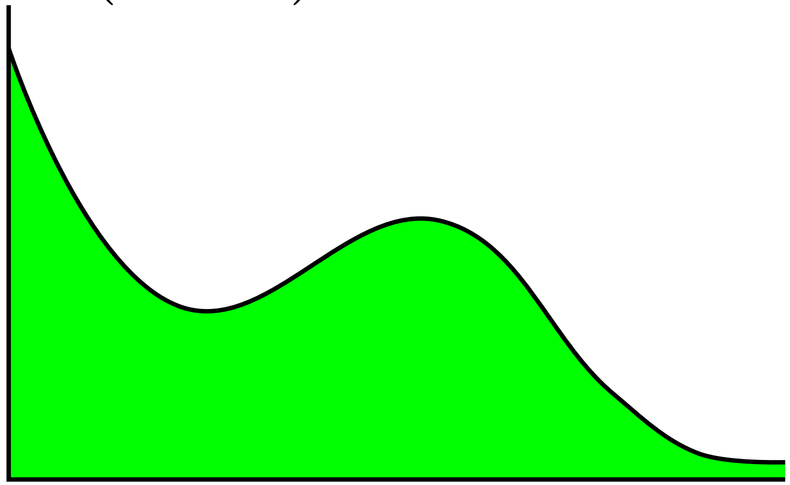
Exposure time : allow particles to exit the domain and re-enter at a later time

$\Rightarrow$

- No BC at the boundary of the control domain
- 'Appropriate' BC at the boundary of the computational domain

Exposure time





$$\bar{\Theta} = \frac{1}{M(t_0)} \int_0^\infty M(t_0+\tau)\,d\tau$$

$=$ Measure of the time $\times$ concentration to which the control region is exposed to particles originating from (t_0, x_0).

$=$ ‘Exposure time’

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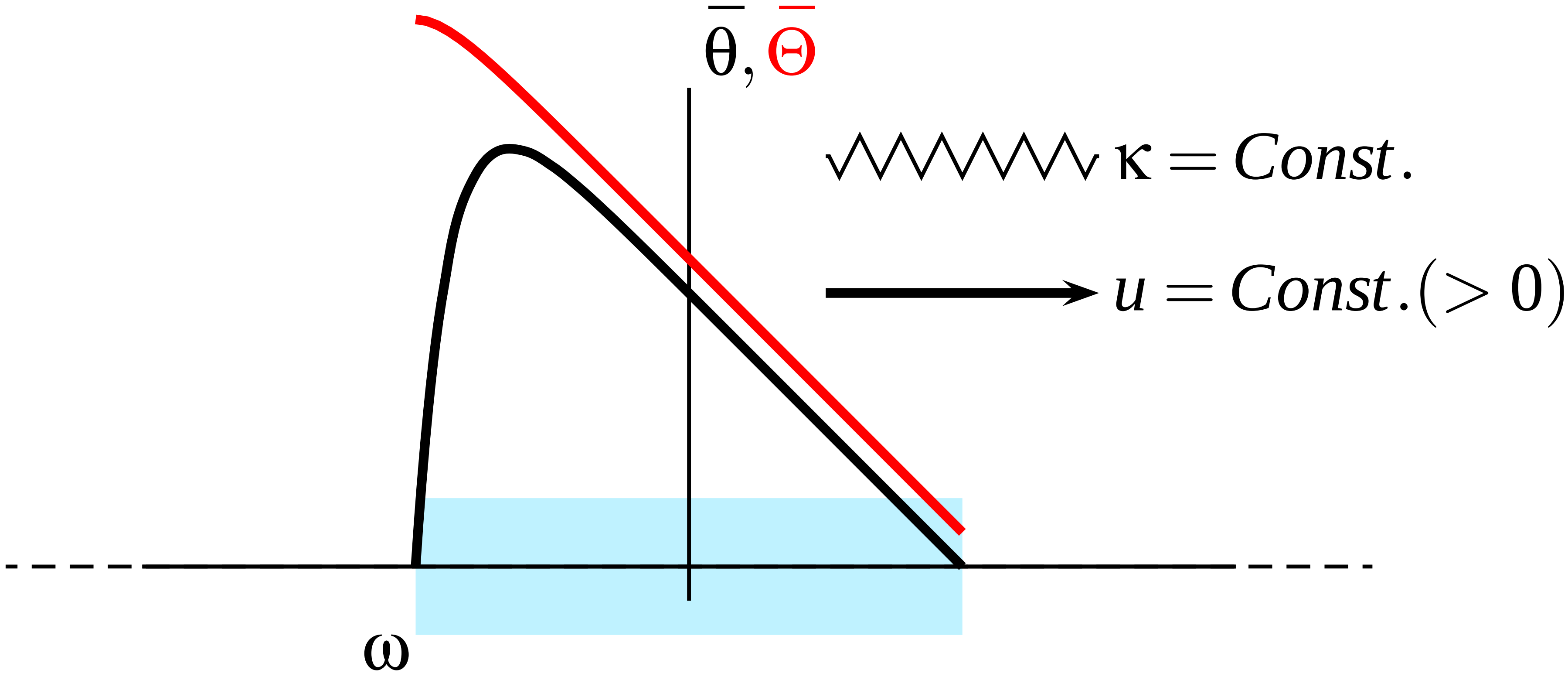
ET : 1D example revisited



$$M(t;t_0,x_0) \geq m(t;t_0,x_0)$$

$$\bar{\Theta}(t,x) \geq \bar{\theta}(t,x)$$

Basin average RT & ET

- Compute

$$\frac{1}{V_\omega}\iiint_\omega \bar{\theta}(x)dx \quad \text{or} \quad \frac{1}{V_\omega}\iiint_\omega \bar{\Theta}(x)dx$$

- or solve

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = \nabla\cdot\left[\mathbf{K}\cdot\nabla C\right] + Q^c \\ C(t_0,x) = 1 \qquad \text{in } \omega \; + \; \text{Approp. B.C.} \end{cases}$$

$$\left.\begin{array}{r} <\bar{\theta}> \\ <\bar{\Theta}> \end{array}\right\} = \frac{1}{V_\omega}\int_{t_0}^{\infty} m(t;t_0,x_0)dt = \frac{1}{V_\omega}\int_{t_0}^{\infty}\iiint_\omega C \; dV dt$$

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Direct Eulerian procedure

Solve

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \nabla \cdot \left[\mathbf{K} \cdot \nabla C \right] \\ C(t_0, x) = \delta(x - x_0) \\ C(t, x) = 0 \text{ on } \partial\omega \end{cases}$$

and compute

$$m(t; t_0, x_0) = \iiint_\omega C \; dV$$

But... multiple runs are needed to compute $\overline{\theta}(t_0, x_0)$.

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Operator formulation

$$m(t;t_0,x_0) = < \mathcal{A}_{t,t_0}\delta(x-x_0), \delta_\omega(x) >$$

where

- $\mathcal{A}_{t,t_0}$ = forward operator such that

$$C(t,x) = \mathcal{A}_{t,t_0}\delta(x-x_0)$$

- δ_ω = characteristic function of control domain ω,

$$\delta_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega, \\ 0 & \text{elsewhere} \end{cases}$$

- $< f,g > = \iiint_{\mathbb{R}^3} f(x)g(x)dV$

Operator formulation (2)

$$m(T;t_0,x_0) = < \mathcal{A}_{T,t_0}\delta(x-x_0), \delta_\omega(x) >$$
$$= < \delta(x-x_0), \mathcal{A}^*_{T,t_0}\delta_\omega(x) >$$

where $\mathcal{A}^*_{T,t_0}$ = adjoint operator of $\mathcal{A}_{T,t_0}$.

$$\begin{cases} -\dfrac{\partial C^*_T}{\partial t} - \mathbf{v}\cdot\nabla C^*_T = \nabla\cdot\left[\mathbf{K}\cdot\nabla C^*_T\right] \\ C^*_T(T,x) = \delta_\omega(x) \\ C^*_T(t,x) = 0 \text{ on } \partial\omega \end{cases}$$

Adjoint state $C^*_T(t_0,x_0) = m(T;t_0,x_0)$

Backward procedure

$$\begin{cases} -\dfrac{\partial C_T^*}{\partial t} - \mathbf{v} \cdot \nabla C_T^* = \nabla \cdot \left[\mathbf{K} \cdot \nabla C_T^* \right] \\ C_T^*(T,x) = \delta_\omega(x), \quad C_T^*(t,x) = 0 \text{ on } \partial\omega \end{cases}$$

- Must be integrated backward in time.
- A single run of the adjoint model provides $m(T;t_0,x_0)$ for a range of (t_0,x_0).

But $m(t_0+\tau;t_0,x_0)$ is required for all $\tau > 0$,
$\Rightarrow$ solve the adjoint problem for a range of 'initial conditions' $C^*_{t_0+\tau}(t_0+\tau,x) = \delta_\omega(x)$ (unless the hydrodynamics is constant)

Backward procedure (2)

Definition : $D(t,\tau,x) = C^*_{t+\tau}(t,x) = m(t+\tau;t,x)$

$$\begin{cases} \dfrac{\partial D}{\partial t} - \dfrac{\partial D}{\partial \tau} + \mathbf{v}\cdot\nabla D + \nabla\cdot\left[\mathbf{K}\cdot\nabla D\right] = 0 \\ D(t,0,x) = \delta_\omega(x), \quad D(t,\tau,x) = \delta(\tau) \text{ on } \partial\omega \end{cases}$$

to be solved in a five-dimensional space.

$\Rightarrow$ Describes the spatial and temporal variations of the cumulative distribution function of residence times $m(t+\tau;t,x)$.

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Mean residence time

$$\bar{\theta}(t,x) = \int_0^\infty m(t+\tau;t,x)\,d\tau = \int_0^\infty D(t,\tau,x)\,d\tau$$

$$\begin{cases} \dfrac{\partial D}{\partial t} - \dfrac{\partial D}{\partial \tau} + \mathbf{v}\cdot\nabla D + \nabla\cdot\left[\mathbf{K}\cdot\nabla D\right] = 0 \\ D(t,0,x) = \delta_\omega(x), \quad D(t,\tau,x) = \delta(\tau) \text{ on } \partial\omega \end{cases}$$

$$\int_0^\infty \ldots d\tau, \text{ assuming } \lim_{\tau\to\infty} D(t,\tau,x) = 0$$

$$\Rightarrow \qquad \begin{cases} \dfrac{\partial\bar{\theta}}{\partial t} + \delta_\omega + \mathbf{v}\cdot\nabla\bar{\theta} + \nabla\cdot\left[\mathbf{K}\cdot\nabla\bar{\theta}\right] = 0 \\ \bar{\theta}(t,x) = 0 \text{ on } \partial\omega \end{cases}$$

Boundary conditions

$$\frac{\partial D}{\partial t} - \frac{\partial D}{\partial \tau} + \mathbf{v} \cdot \nabla D + \nabla \cdot \left[\mathbf{K} \cdot \nabla D \right] = 0$$

or

$$\frac{\partial \overline{\theta}}{\partial t} + \delta_\omega + \mathbf{v} \cdot \nabla \overline{\theta} + \nabla \cdot \left[\mathbf{K} \cdot \nabla \overline{\theta} \right] = 0$$

At open boundaries of the control region ω prescribe

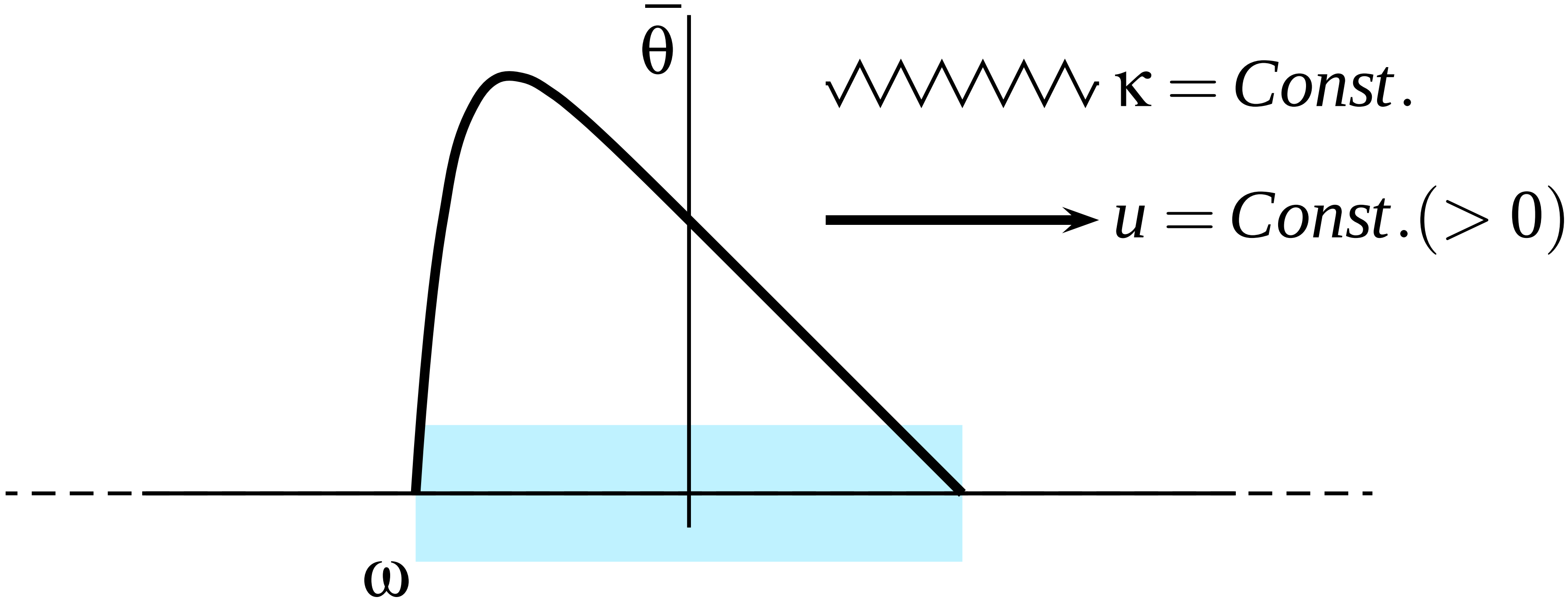
- $D(t,\tau,x) = \delta(\tau - 0)$

or

- $\overline{\theta}(t,x) = 0$

$\Rightarrow$ Residence time in ω

1D Exemple - Residence time



$$\frac{\partial(\cdot)}{\partial t} = 0$$

$$\begin{cases} \delta_{]-L,L[} + u\frac{d\overline{\theta}}{dx} + \kappa\frac{d^2\overline{\theta}}{dx^2} = 0 \\ \overline{\theta}(+L) = \overline{\theta}(-L) = 0 \end{cases}$$

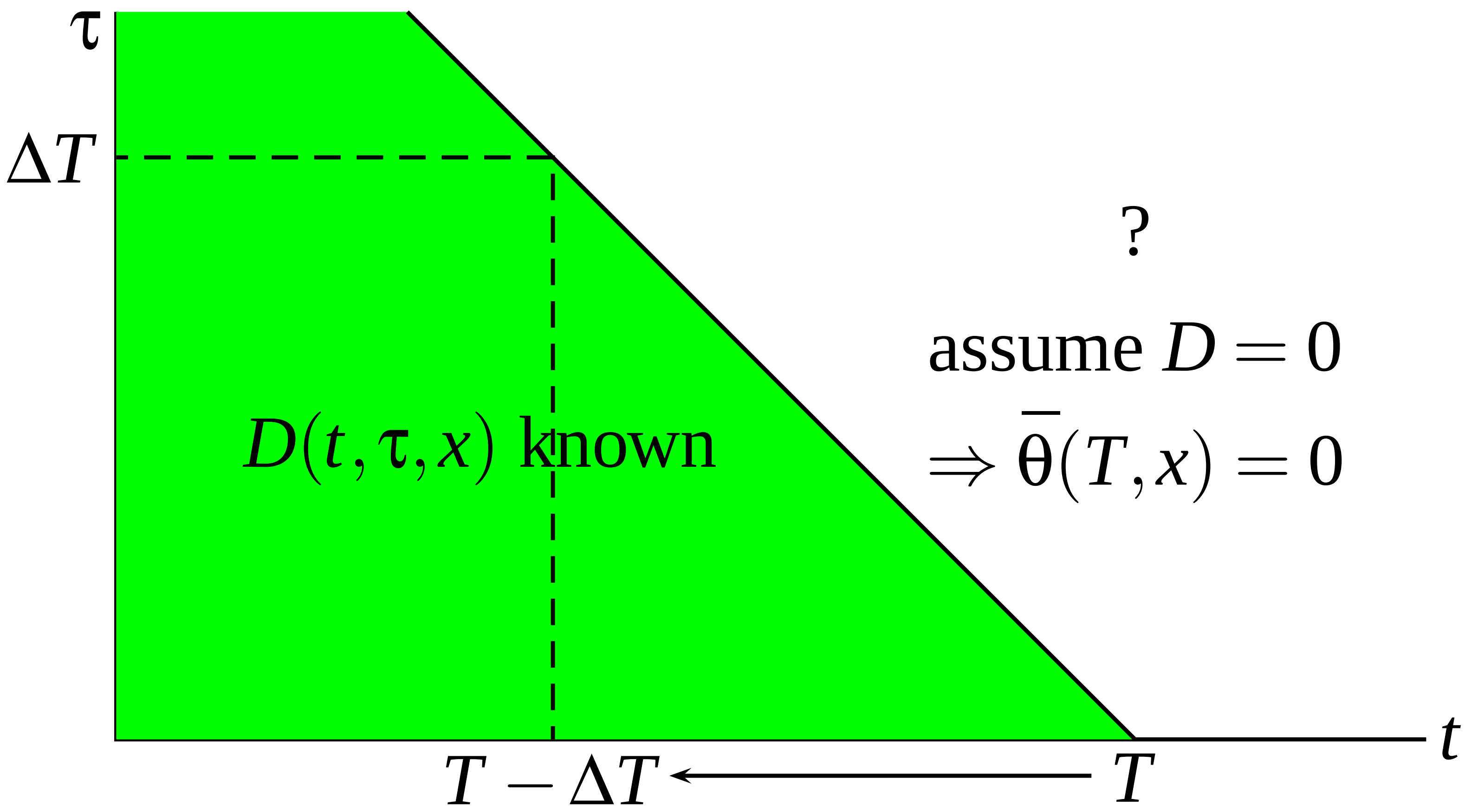
Initial conditions

$$\frac{\partial\overline{\theta}}{\partial t} + \delta_\omega + \mathbf{v}\cdot\nabla\overline{\theta} + \nabla\cdot\left[\mathbf{K}\cdot\nabla\overline{\theta}\right] = 0$$

Must be solved by backward integration from ‘initial conditions’ given at some time T.

$$\overline{\theta}(T,x) = ?$$

Initial conditions (2)



$$\bar{\theta}(t,x) = \int_0^\infty D(t,\tau,x)\,d\tau = \int_0^T D(t,\tau,x)\,d\tau$$

only material with RT < ΔT is taken into account

Initial conditions (3)

Concentration of material with RT < ΔT is given by

$$\tilde{C}_T = 1 - C_T^*$$

where

$$\begin{cases} -\dfrac{\partial C_T^*}{\partial t} - \mathbf{v} \cdot \nabla C_T^* = \nabla \cdot \left[\mathbf{K} \cdot \nabla C_T^* \right] \\ C_T^*(T, x) = \delta_\omega(x) \end{cases}$$

Summary

$$\begin{cases} \dfrac{\partial \overline{\theta}}{\partial t} + \delta_\omega + \mathbf{v} \cdot \nabla \overline{\theta} + \nabla \cdot \left[\mathbf{K} \cdot \nabla \overline{\theta} \right] = 0 \\ \overline{\theta}(T, x) = 0, \;\; \overline{\theta}(t, x) = 0 \text{ on } \partial\omega \end{cases}$$

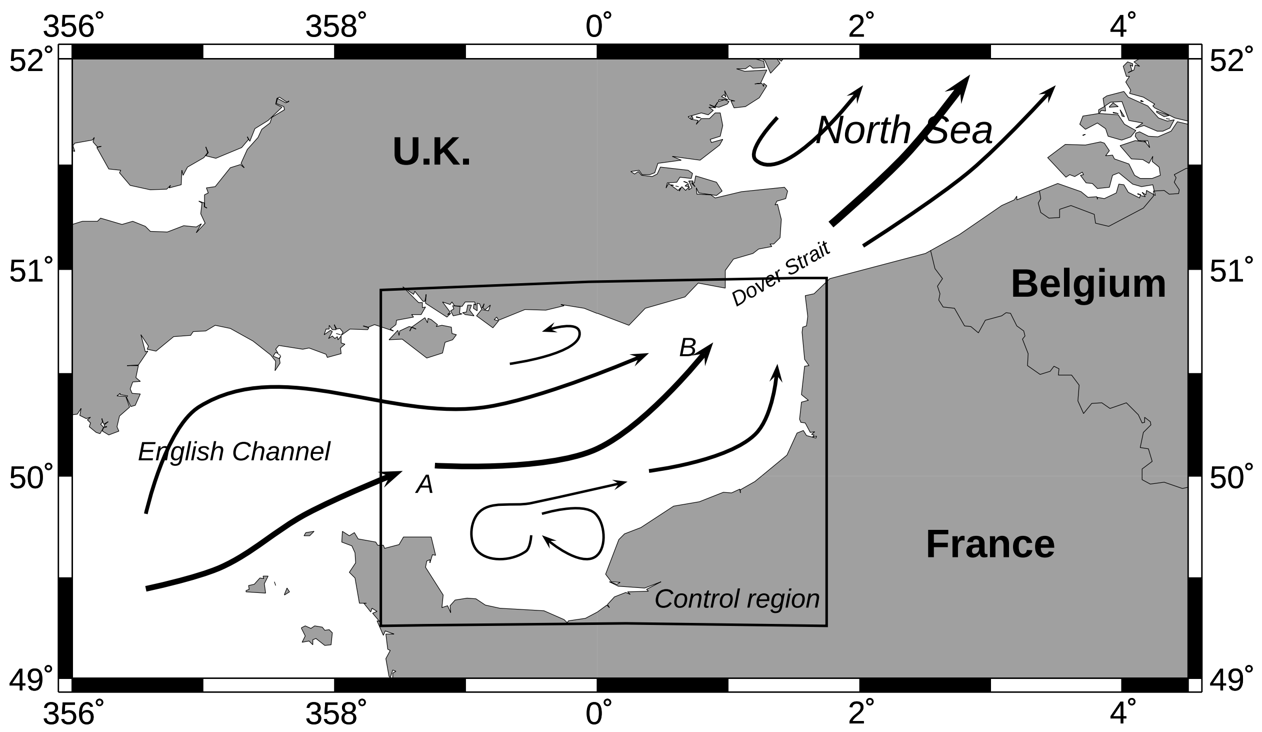
$\Rightarrow$ mean residence time

$$\begin{cases} \dfrac{\partial C_T^*}{\partial t} + \mathbf{v} \cdot \nabla C_T^* + \nabla \cdot \left[\mathbf{K} \cdot \nabla C_T^* \right] = 0 \\ C_T^*(T, x) = \delta_\omega(x), \;\; C_T^*(t, x) = 0 \text{ on } \partial\omega \end{cases}$$

$\Rightarrow$ convergence if $\tilde{C}_T = 1 - C_T^* \approx 1$

NWECS model

- $\Delta_x = \Delta_z = 10'$, 10 σ-levels
- free-surface, baroclinic, k turbulence model
- 10 tidal constituents, NCEP Reanalysis met. data



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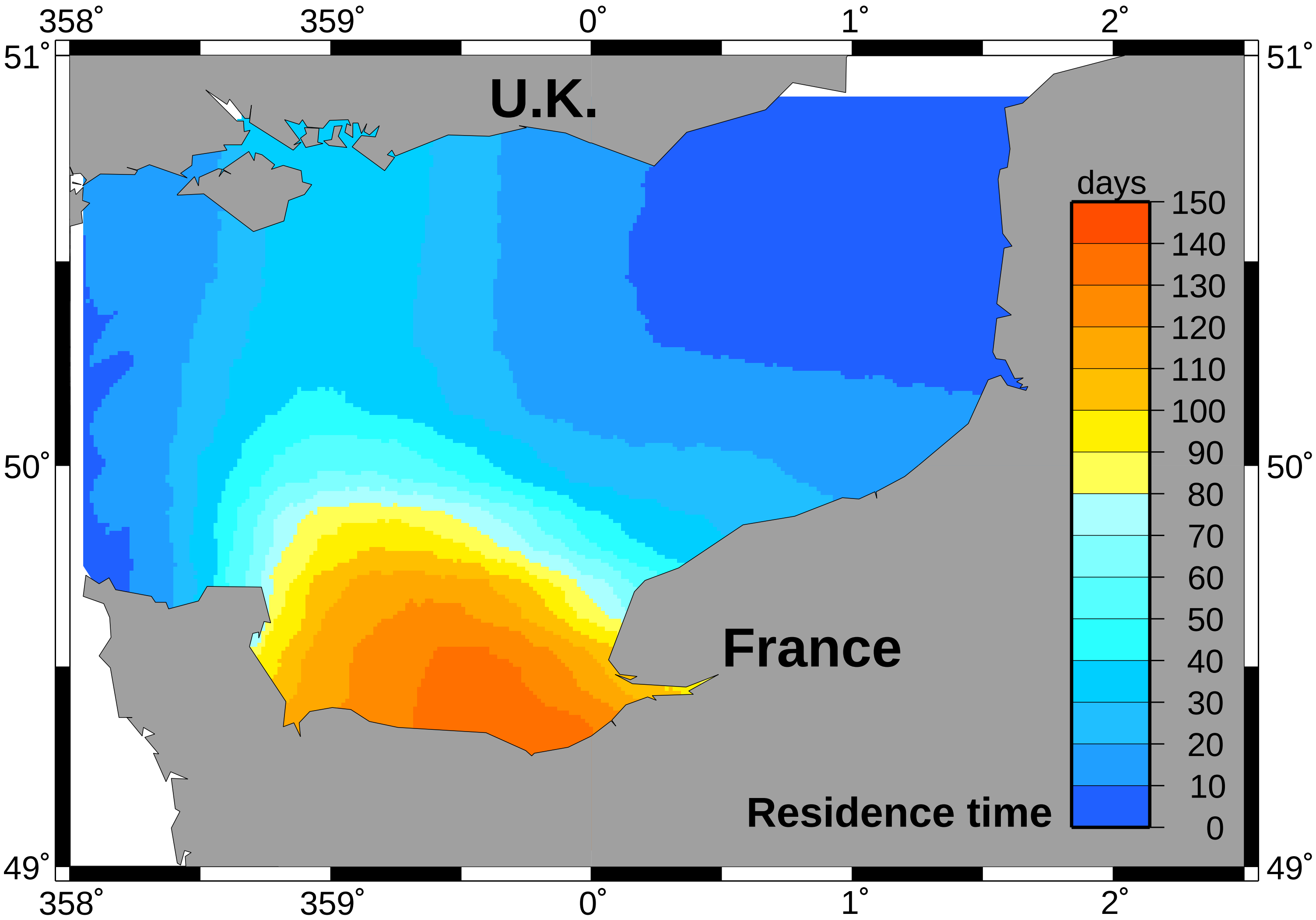
Residence time (days)



Snapshot on 15/08/1983 - Surface value

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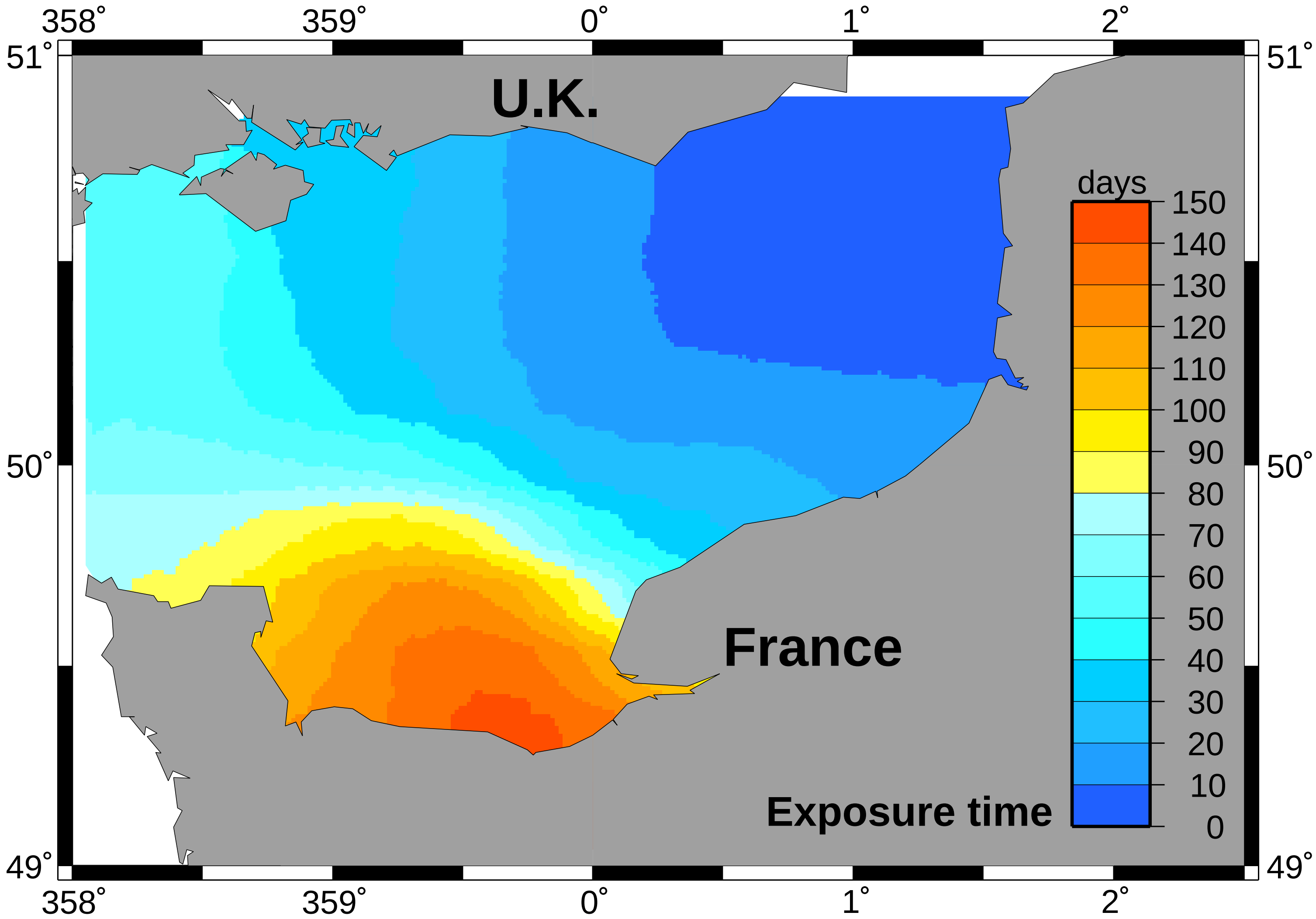
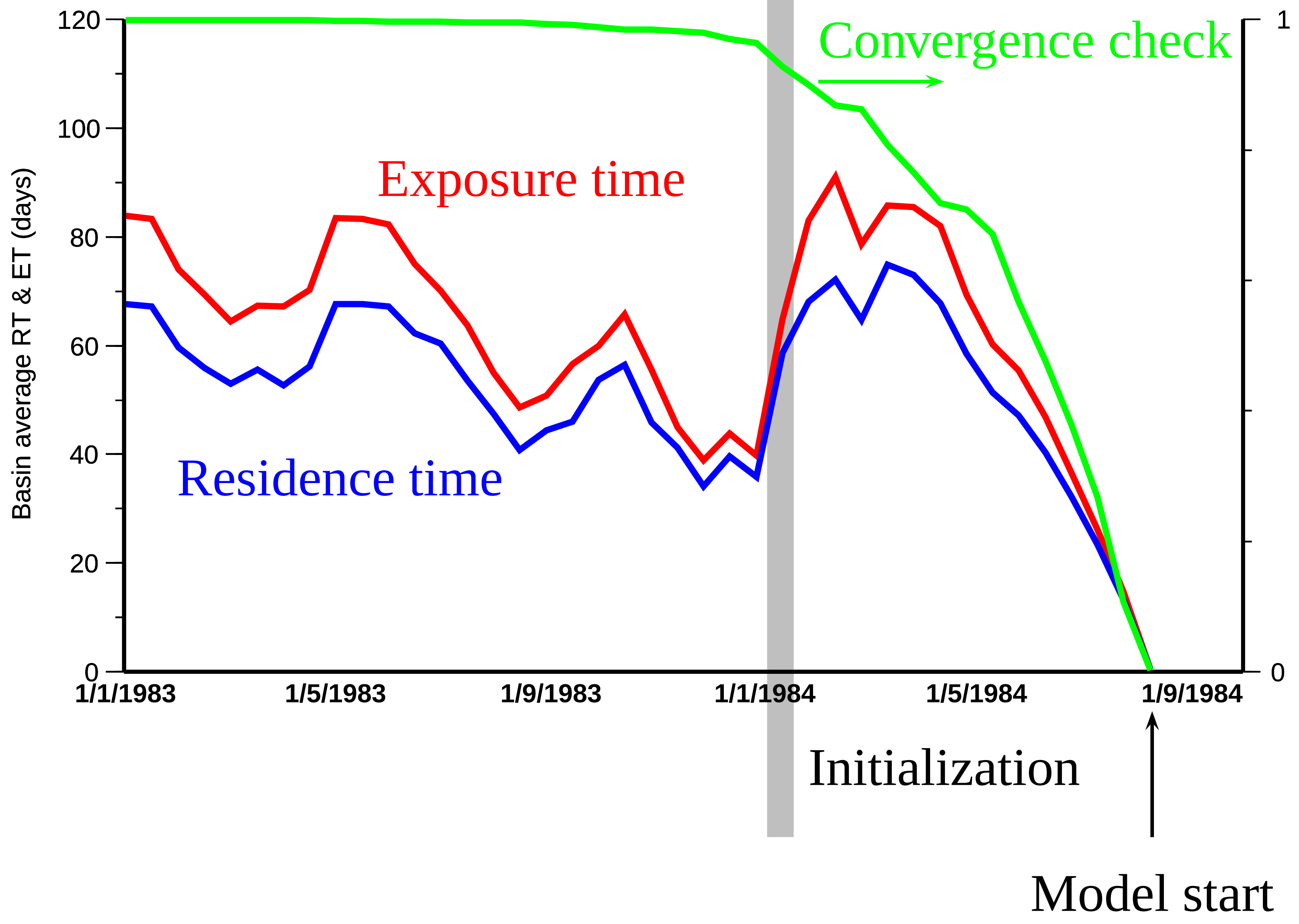
Exposure time (days)



Snapshot on 15/08/1983 - Surface value

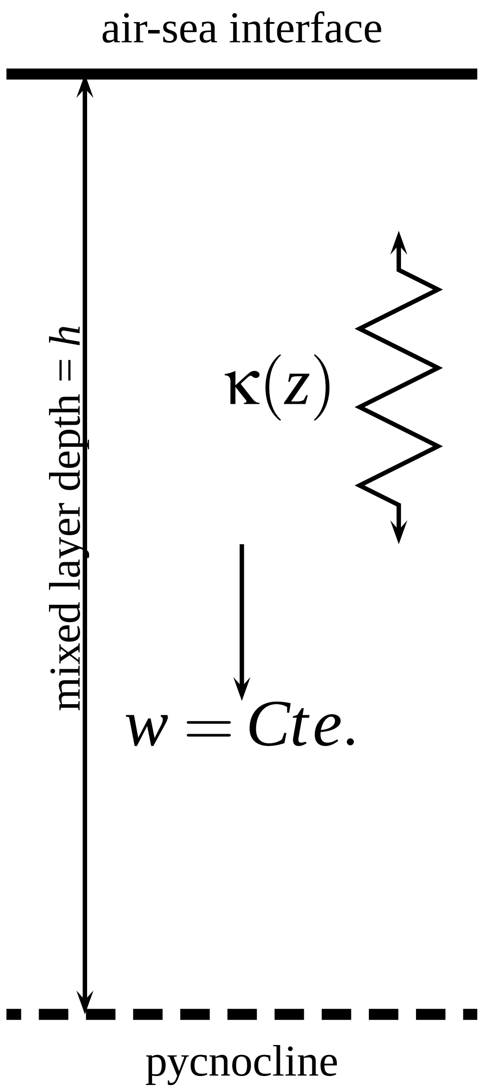
Variability



RT in the mixed layer

Does turbulence increase or decrease the residence time in the surface mixed layer of settling particles ?

RT in the mixed layer : set-up



$$wC + \kappa \frac{\partial C}{\partial z} = 0 \qquad \kappa \frac{\partial \theta}{\partial z} = 0$$

$$\begin{cases} \dfrac{\partial C}{\partial t} = \dfrac{\partial}{\partial z}\left(wC + \kappa(z) \dfrac{\partial \theta}{\partial z} \right) \\ C(0,z) = \delta(z - z_0) \end{cases}$$

$$\frac{d}{dz}\left(w\theta - \kappa(z) \frac{d\theta}{dz} \right) = 1$$

$$\kappa \frac{\partial C}{\partial z} = 0 \qquad wC - \kappa \frac{\partial \theta}{\partial z} = 0$$

RT in the mixed layer : results

$$\theta(z) = \frac{z}{w} + \frac{1}{w}\int_z^h \exp\left[-w\int_z^\zeta \frac{d\zeta}{\kappa(\zeta)}\right] d\zeta$$

$$\frac{z}{w} \quad < \quad \theta(z) \quad < \quad \frac{h}{w}$$

No diffusion — Infinite mixing

$$\frac{1}{2}\frac{h}{w} \quad < \quad \bar{\theta} = \frac{1}{h}\int_0^h \theta(z)dz \quad < \quad \frac{h}{w}$$

Factor 2 only !

Residence time increases with turbulence !

Conclusion of part II

- Useful diagnostic for numerical models
- Flexibility : residence / exposure time
- Can be generalized to tracers with linear dynamics.

http://www.climate.be/CART